Evaluation of stable periodic orbits about non-spherical objects

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Outline

- Introduction & Motivation
- Background
  - Controls
  - Optimization
- Methodology
  - Dynamic Environment Model
  - Gravitational Potential Field Model
  - Propagator Algorithm
  - Periodic Orbit Solvers
- Simulation and Results
  - 433 Eros
  - Chaos
  - Orbit Analysis
- Ongoing Efforts
- Conclusions / Lessons Learned

Introduction & Motivation

- Non-spherical objects with non-uniform mass distribution and density
  - Guidance, Navigation, and Control (GNC) of spacecraft
    - Future missions to asteroids: small orbiting satellite(s)?
      - OSIRUS-Rex
    - Humans to asteroids: orbiters?
  - Origin studies
- *Why are periodic or quasi-periodic orbits interesting?*

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Background: Controls

- Poincare-Bendixson Theorem[1] suggests that a compact domain D, *excluding equilibrium points*, with a vector field pointing towards its interior will have at least one stable, periodic trajectory (i.e. limit cycle).

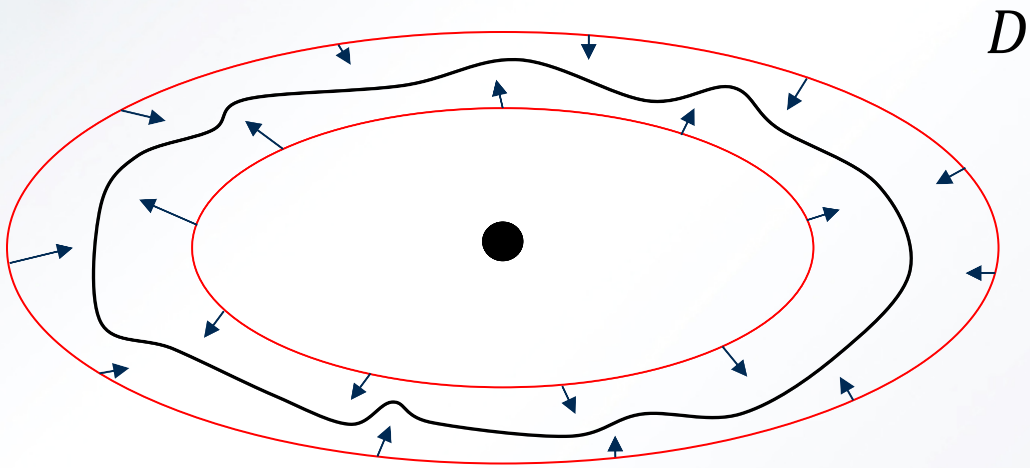

Figure 1: Poincare-Bendixson Theory

- Imagine what this implies for an n dimensional space, namely when $n = 3$.

Background: Optimization

- Constrained minimization problem:

$$\begin{aligned} &\min f(x), \forall x \in D \\ &s.t. \quad g(x) \le 0 \\ &\qquad\quad h(x) = 0 \end{aligned}$$

- In this problem, we aim to ultimately solve:

$$\begin{aligned} &\min \qquad J = \frac{1}{2}\int_{t_0}^{t_f} \boldsymbol{u}(\tau)^2 d\tau \\ &s.t. \\ &\boldsymbol{x_B}(t_0) = \boldsymbol{x}_A(t_0) \\ &\boldsymbol{x_B}(t_f) = \boldsymbol{x}_C(t_f) \\ &\boldsymbol{u}(t) \le u_{max} \\ &\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t) \end{aligned}$$

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Background: Optimization

- When worst comes to worst, run a Monte-Carlo simulation
- Begin with a quasi-stable orbit and adjust initial conditions according to:

$$\boldsymbol{x}_{\boldsymbol{k}+\mathbf{1}}(t_0) = \boldsymbol{x}_{\boldsymbol{k}}(t_0) + \boldsymbol{\delta x}_{\boldsymbol{k}}(t_0)$$

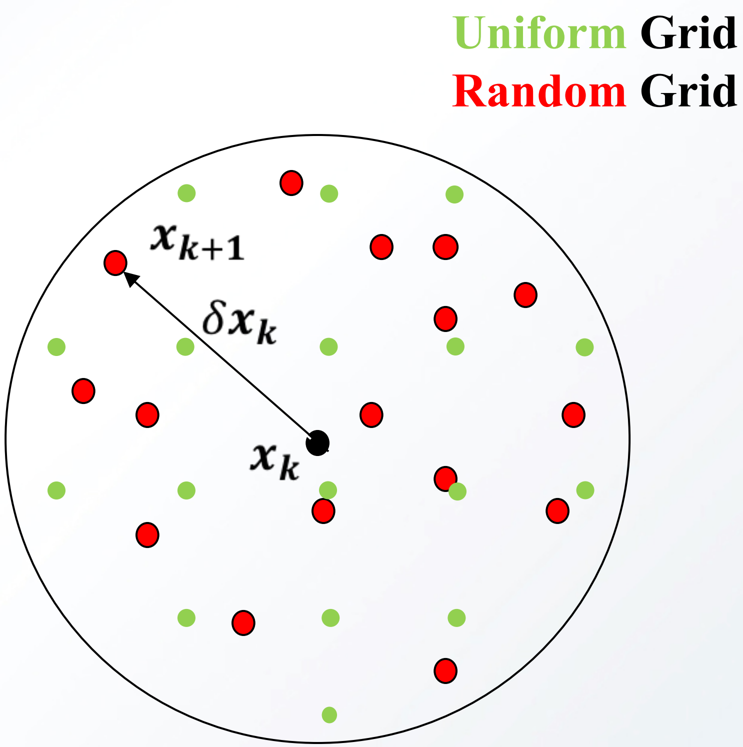

Figure 2: Randomization of Initial Condition Vectors

Methodology: Dynamic Environment Model

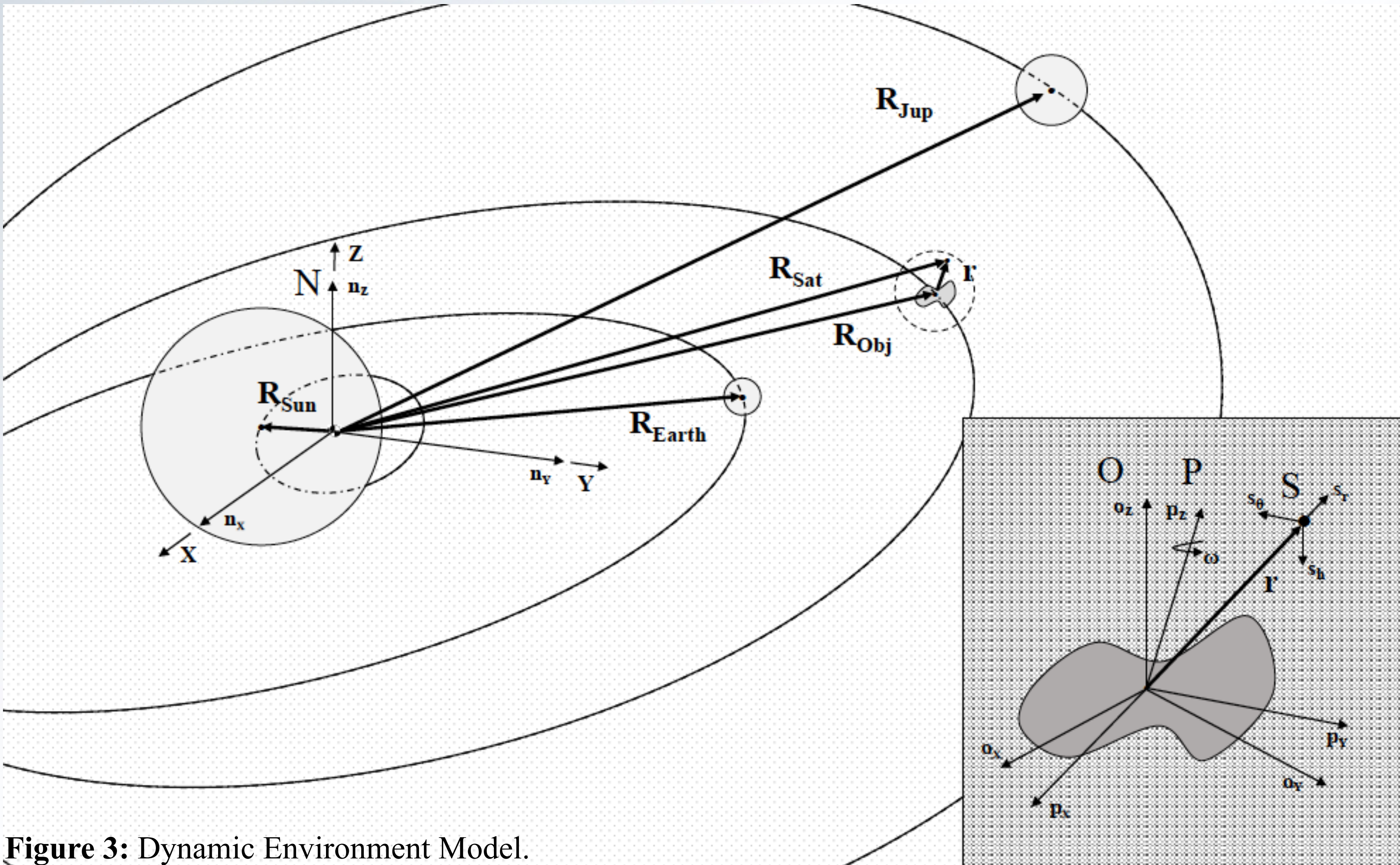

Figure 3: Dynamic Environment Model.

Methodology: Dynamic Environment Model

$$\ddot{\boldsymbol{r}}_i = -\frac{\mu}{\|\boldsymbol{r}_i\|^3}\boldsymbol{r}_i + \boldsymbol{a}_{Pert}$$

$$\boldsymbol{a}_{Pert} = \boldsymbol{a}_{srp} + \boldsymbol{a}_{3rd} + \boldsymbol{a}_{MacCullagh}$$

$$\boldsymbol{a}_{srp} = \left[\frac{A(1+\rho)G^*}{m\left\|\boldsymbol{R}_{obj} - \boldsymbol{R}_{Sun}\right\|^2}\right]\left(\boldsymbol{R}_{obj} - \boldsymbol{R}_{Sun}\right)\cdot \boldsymbol{r}$$

"Cannonball" Method[2]

$$\boldsymbol{a}_{3rd} = \mu_j\left[\frac{\boldsymbol{R}_j - \boldsymbol{R}_i}{\left\|\boldsymbol{R}_j - \boldsymbol{R}_i\right\|^3} - \frac{\boldsymbol{R}_j}{\left\|\boldsymbol{R}_j\right\|^3}\right]$$

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Methodology: Dynamic Environment Model

$$\ddot{\boldsymbol{r}}_i = -\frac{\mu}{\|\boldsymbol{r_i}\|^3}\boldsymbol{r}_i + \boldsymbol{a}_{Pert}$$

$$\boldsymbol{a}_{Pert} = \boldsymbol{a}_{srp} + \boldsymbol{a}_{3rd} + \boldsymbol{a}_{MacCullagh}$$

$$\boldsymbol{a}_{srp} = \left[\frac{A(1+\rho)G^*}{m\|\boldsymbol{R_{obj}} - \boldsymbol{R_{Sun}}\|^2}\right](\boldsymbol{R_{obj}} - \boldsymbol{R_{Sun}}) \cdot \boldsymbol{r}$$

"Cannonball" Method[2]

$$\boldsymbol{a}_{3rd} = \mu_j\left[\frac{\boldsymbol{R}_j - \boldsymbol{R}_i}{\|\boldsymbol{R_j} - \boldsymbol{R_i}\|^3} - \frac{\boldsymbol{R}_j}{\|\boldsymbol{R_j}\|^3}\right]$$

Third Body Forces

Methodology: Gravitational Potential Field Model

- MacCullagh's Approximation[3]:

$$\boldsymbol{a}_{MacCullagh} = -\frac{Gm}{\|\boldsymbol{r}\|^2}\widehat{\boldsymbol{s}_r} - \frac{3G}{2\|\boldsymbol{r}\|^4}\{I_{xx} + I_{yy} + I_{zz} - 5I_r\}\widehat{\boldsymbol{s}_r} + \frac{3G}{2\|\boldsymbol{r}\|^5}\{I_{xx}\widehat{\boldsymbol{o}_x} + I_{yy}\widehat{\boldsymbol{o}_y} + I_{zz}\widehat{\boldsymbol{o}_z}\}$$

Point-mass contribution

Polar moment of inertia contribution

Moment of inertia contribution

- Eliminates the need for spherical harmonic coefficients at the expense of computational accuracy
- Appears to provide reasonable results for a first-order analysis of objects about non-spherical objects

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Methodology: Propagator Algorithm

Algorithm 1 Hybrid Propagator using STMs and GVEs

procedure Hybrid Propagator for Satellite about Body-of-Interest
  set e_{tol}
  set i_{tol}
  set t_{vec}
  for $k = 1 \to \text{length}(t_{vec})$
    convert $\{\mathbf{r_k}, \dot{\mathbf{r}}_\mathbf{k}\} \to \{a_k, e_k, i_k, \Omega_k, \omega_k, M_k\}$
    if $e_k \geq e_{tol}$ **or** $i_k \geq i_{tol}$ **then**
      pass $\{t_k, a_k, e_k, i_k, \Omega_k, \omega_k, M_k\} \to$ **odeGVE.m**
      compute $\frac{da}{dt}, \frac{de}{dt}, \frac{di}{dt}, \frac{d\Omega}{dt}, \frac{d\omega}{dt}$, and $\frac{dM}{dt}$
      exit **odeGVE.m**
      get $\{a_{k+1}, e_{k+1}, i_{k+1}, \Omega_{k+1}, \omega_{k+1}, M_{k+1}\}$
      convert $\{a_{k+1}, e_{k+1}, i_{k+1}, \Omega_{k+1}, \omega_{k+1}, M_{k+1}\} \to \{\mathbf{r_{k+1}}, \mathbf{v_{k+1}}\}$
    else
      pass $\{t_k, r_{x,k}, r_{y,k}, r_{z,k}, \dot{r}_{x,k}, \dot{r}_{y,k}, \dot{r}_{z,k}\} \to$ **odeSTM.m**
      compute $A = \begin{bmatrix} Z_{3x3} & I_{3x3} \\ J_{3x3} & Z_{3x3} \end{bmatrix}$
      compute $\dot{\mathbf{x}}(t) = A(t)\mathbf{x} + B(t)\mathbf{u}$
      exit **odeSTM.m**
      get $\{\mathbf{r_{k+1}}, \dot{\mathbf{r}}_\mathbf{k+1}\}$
    end
  end

Figure 4: Hybrid Propagator using STMs and GVEs.

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Methodology: Periodic Orbit Solvers

- Desire: $\boldsymbol{x}(t_0) = \boldsymbol{x}(t_0 + NT)$
- One method[4] is to update state vector according to: $\boldsymbol{x}_{\boldsymbol{k}+\mathbf{1}}(t_0) = \boldsymbol{x}_{\boldsymbol{k}}(t_0) + \Phi_k(t, t_0)^{-1}[\boldsymbol{x}_{\boldsymbol{k}}(t) - \boldsymbol{x}_{\boldsymbol{k}}(t_0)]$
  - ***Highly*** sensitive to the S.T.M. Many times the solution $\boldsymbol{x}_{\boldsymbol{k}+\mathbf{1}}(t_0)$ diverges with no hope of landing within reason again.
  - Better with small time steps
- Another method is scan subspace of $\mathbb{R}^{\mathbf{3}}$ s.t. $\boldsymbol{x}_{\boldsymbol{k}+\mathbf{1}}(t_0) = \boldsymbol{x}_{\boldsymbol{k}}(t_0) + \boldsymbol{\delta x}_{\boldsymbol{k}}(t_0)$
  - This method works ***surprisingly well*** (at the expense of computational time, of course)

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Simulation and Results: 433 Eros

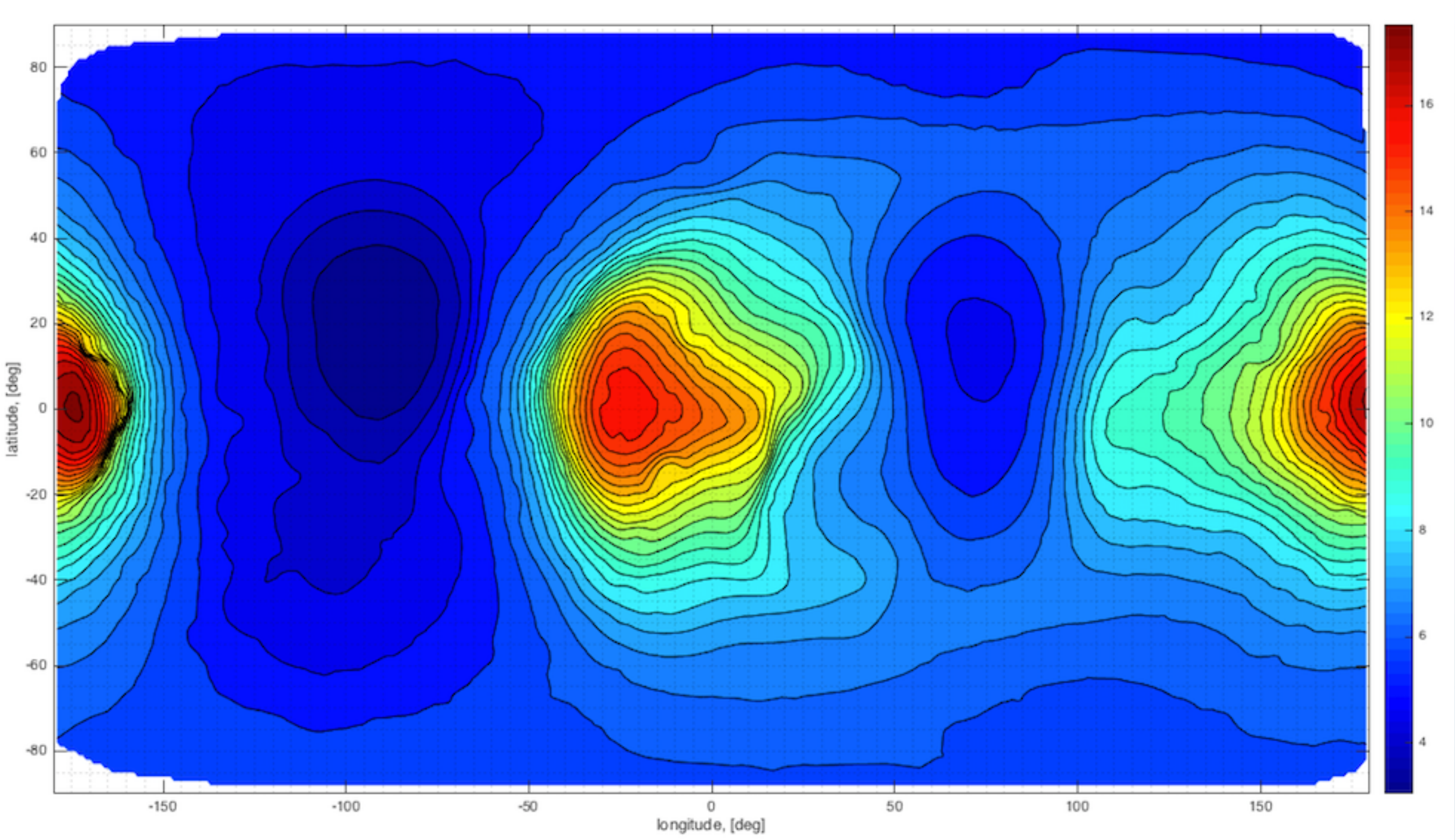
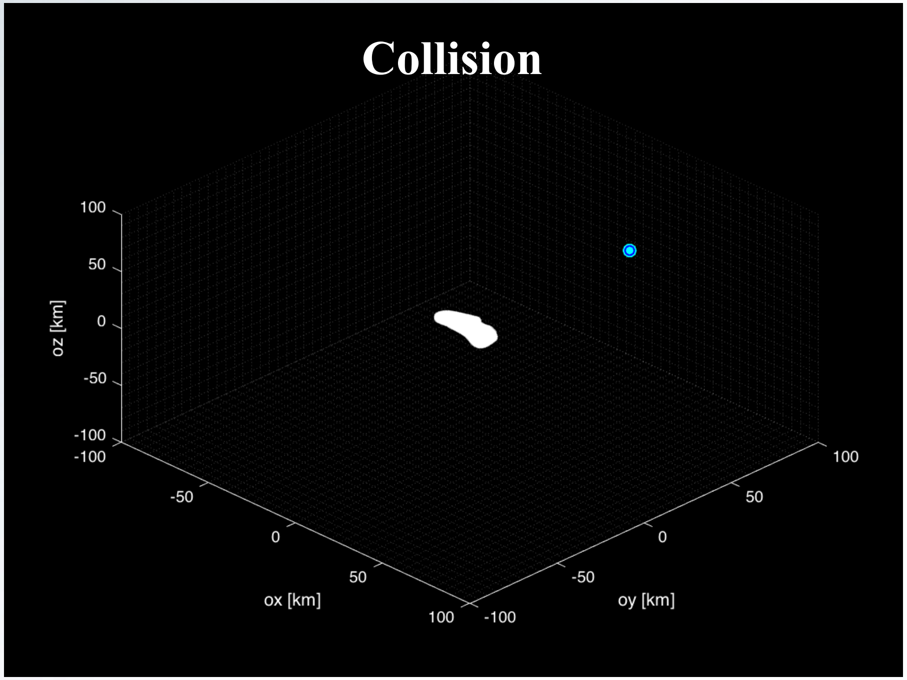


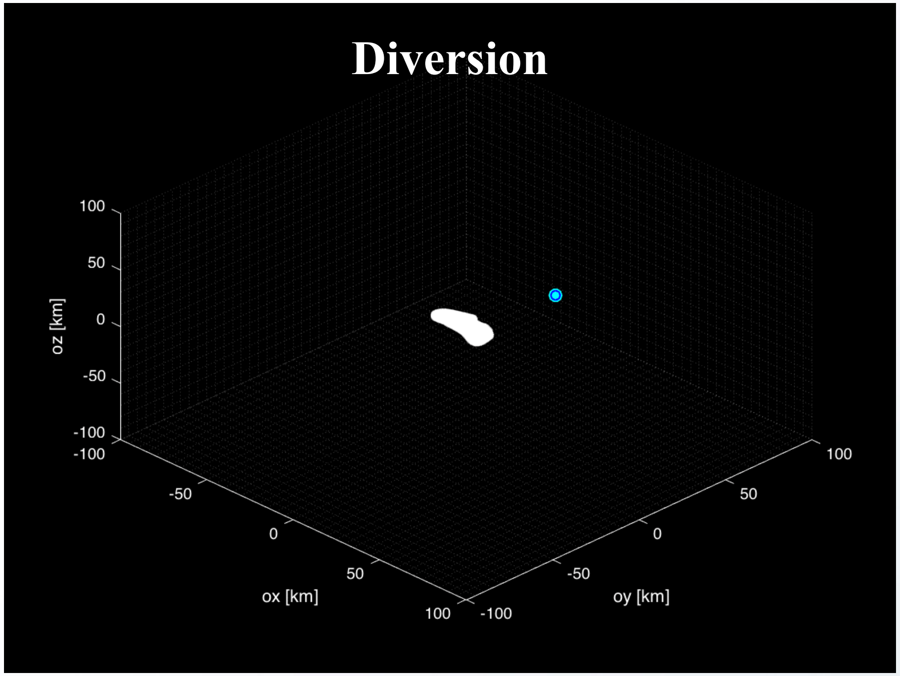
Figure 5: 433 Eros Radii Contour Map. Shape model adapted from Gaskell[5].

Simulation and Results: Chaos

- Chaotic orbits are **common** about the non-spherical object.



(a)



(b)

Figure 6: (a) Unstable Collision Trajectory, (b) Unstable Diverging Trajectory.

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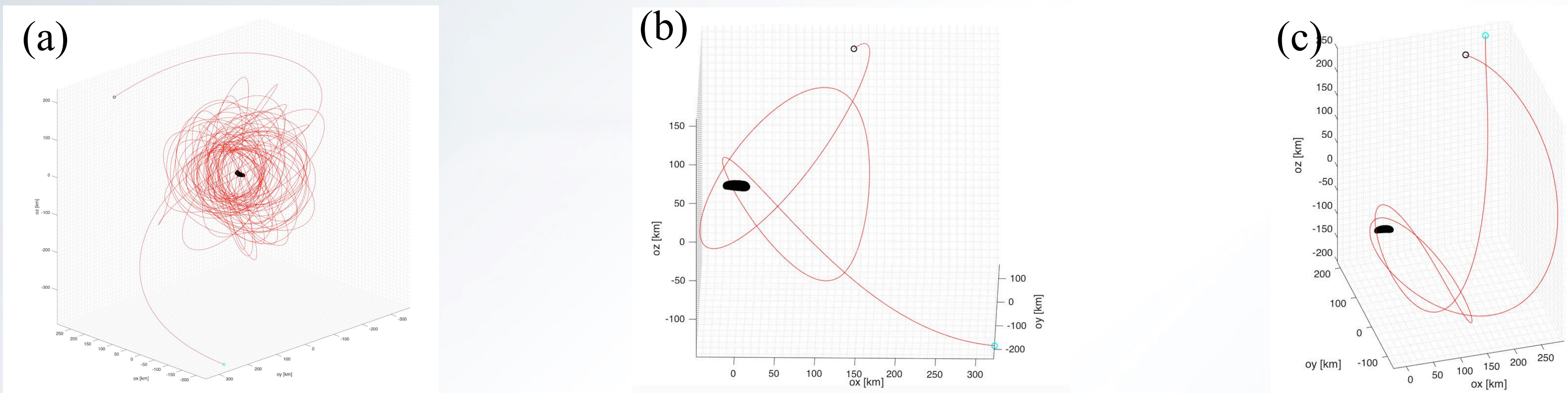
Simulation and Results: Chaos

- Sometimes the spacecraft orbit diverges…



- And sometimes the orbit crashes into the body…

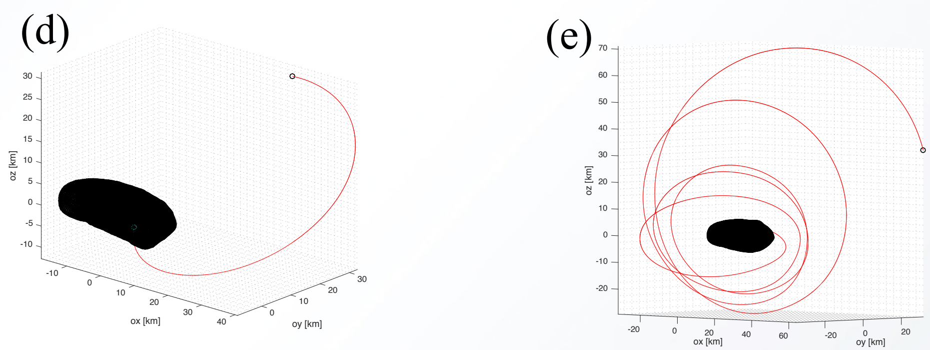


Figure 7: (a) - (c) Diverging Trajectories, (d) - (e) Collision Trajectories.

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Simulation and Results: Orbit Analysis

- SRP, Third Body, and Gravity Field Approximation

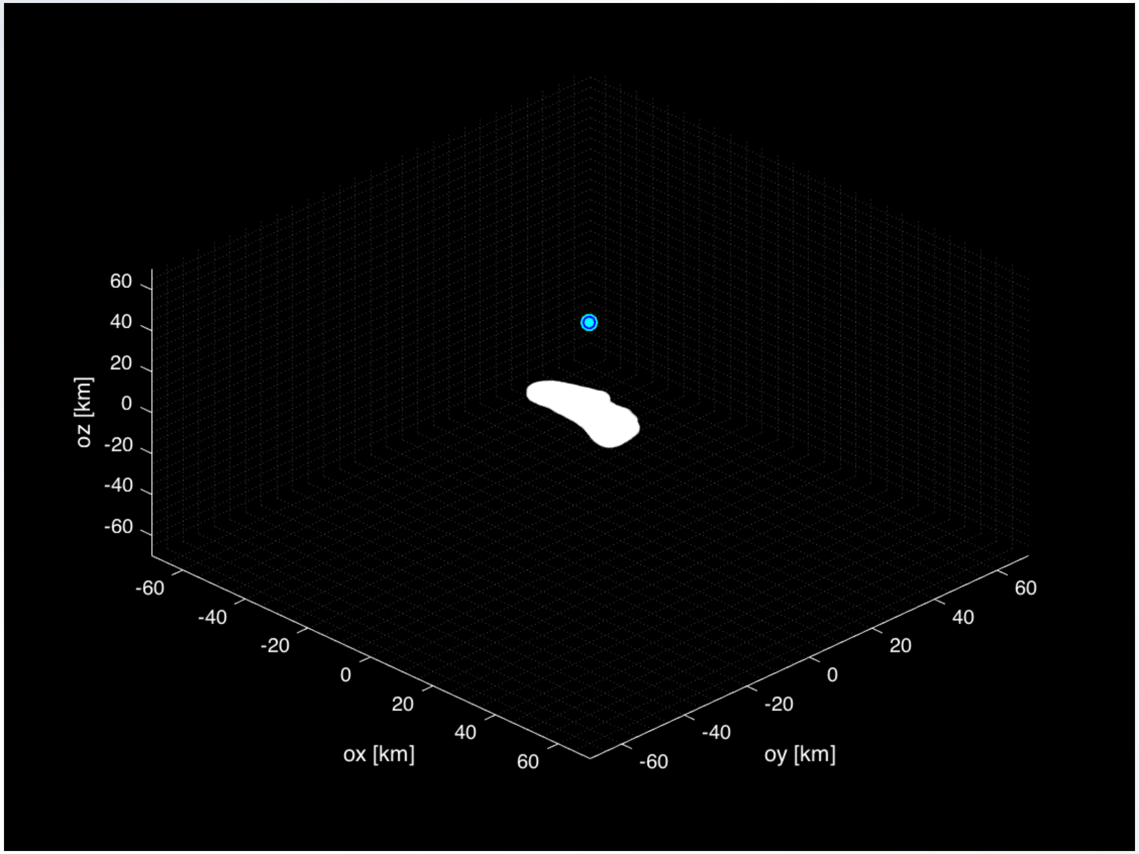

Figure 8: Stable, Quasi-Periodic Terminator Orbit about 433 Eros.

Simulation and Results: Orbit Analysis

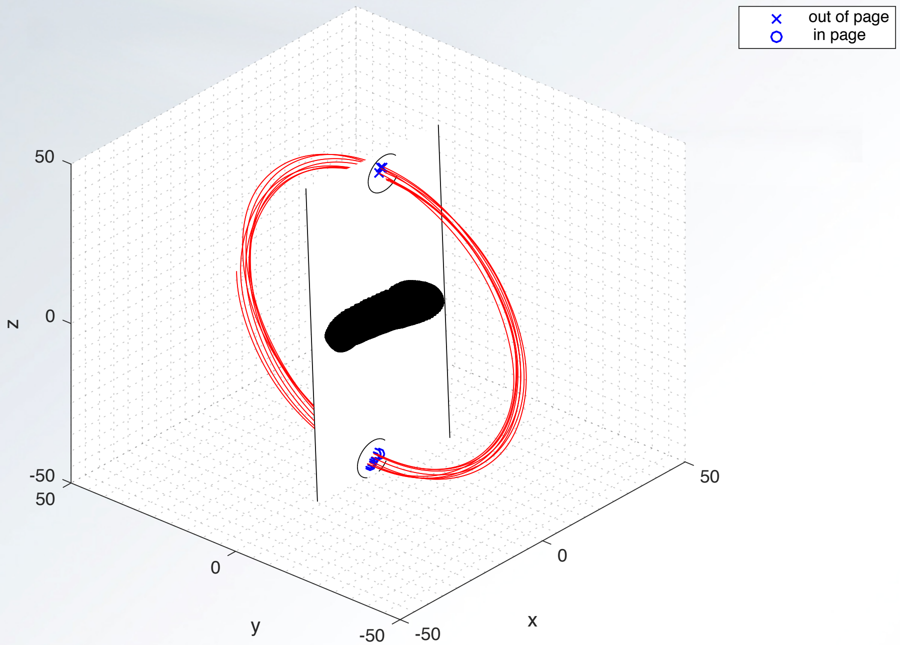

Figure 9: 3D View with Poincare Plane.

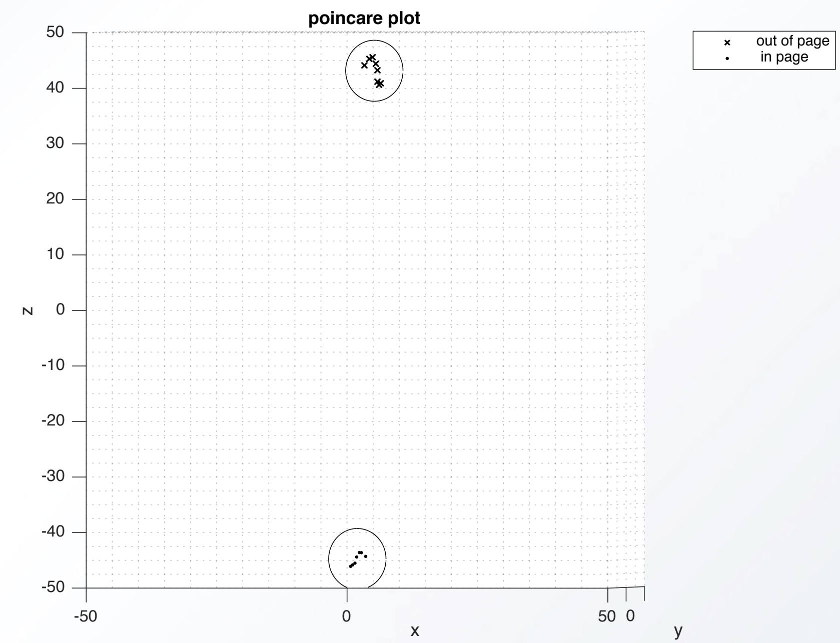

Figure 10: Poincare Plot.

Simulation and Results: Orbit Analysis

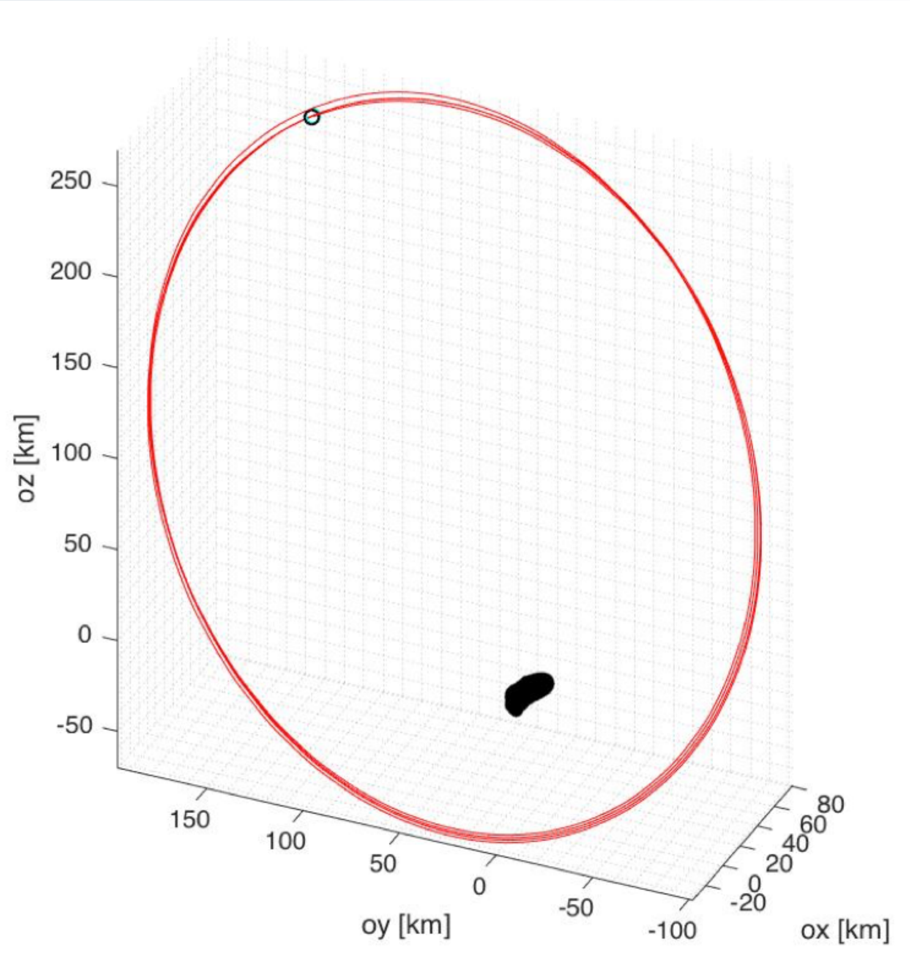

Figure 11: Another Terminator Orbit.

- MacCullagh approximation yields two distinct terminator orbits of varying eccentricities, both stable, quasi-periodic.
- Scheeres et. al. predicted these to be unstable orbit solutions about 433 Eros, using a complete set of equations of motion.

Simulation and Results: Orbit Analysis

- What is going on here?
- MacCullagh's approximation is *still* valid
- Limitations in MacCullagh's approximation involved with apriori knowledge of only the moment of inertia matrix.
- Existing force models

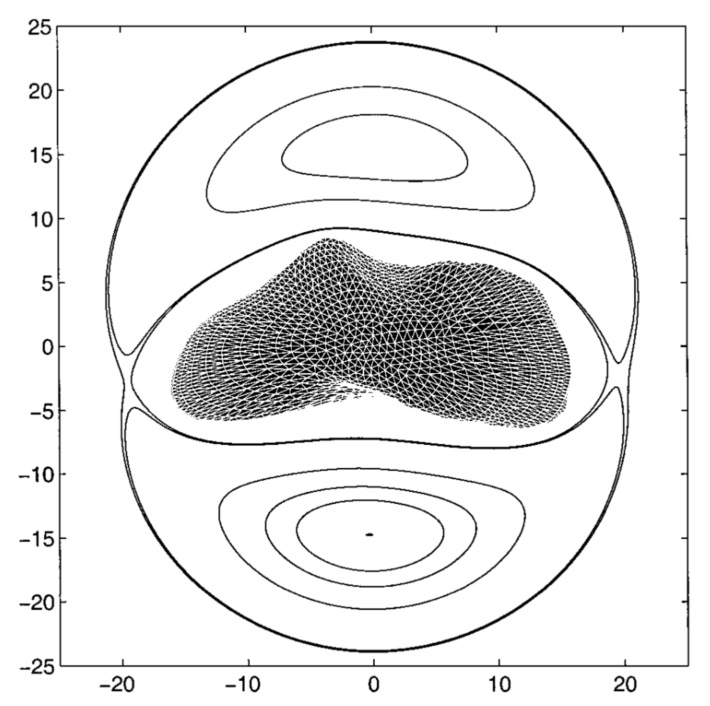

Figure 12: Scheeres' Zero-Velocity Plot.[6]

Ongoing Efforts

- Revisit current gravitational potential field model
  - Is there a method for approximating spherical harmonic coefficients?
- Solve the two-point boundary problem with variable specific impulse using SQP:

$$
\begin{aligned}
\min \quad & J = \frac{1}{2}\int_{t_0}^{t_f} \boldsymbol{u}(\tau)^2 d\tau \\
s.t. \quad & \boldsymbol{x}_{\boldsymbol{B}}(t_0) = \boldsymbol{x}_A(t_0) \\
& \boldsymbol{x}_{\boldsymbol{B}}(t_f) = \boldsymbol{x}_C(t_f) \\
& \boldsymbol{u}(t) \leq u_{max} \\
& \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t)
\end{aligned}
$$

Conclusions / Lessons Learned

- Results are only as good as the model approximation.
- Orbit approximations are only good for ~7 days.
- Addition of perturbations yield quasi-periodic orbits.
- Apriori information increases model fidelity.

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Questions

Thank you for your attention.

References

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- [2] D.A. Vallado. *Fundamentals of Astrodynamics and Applications.* Microcosm Press and Kluwer Academic Publishers, El Segundo, CA, USA, 2001.
- [3] H. Schaub and J.L. Junkins. *Analytical Mechanics of Space Systems.* AIAA Education Series. 2003.
- [4] Scheeres, D.J. *Dynamics of Asteroids close to 4179 Toutatis.* Icarus. Vol. 132, No. 1, p. 53 – 79, 1998.
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- [6] Williams B.G., Scheeres, D.J., and J.K. Miller. *Evaluation of the dynamics environment of an asteroid: Applications to 433 Eros.* Journal of Guidance, Control, and Dynamics. Vol. 23, No. 3, p. 466 – 475, 2000.

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